

Velocity operator approach to quantum fluid dynamics in a three-dimensional neutron-proton system

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Abstract

In the preceeding paper, introducing *isospin*-dependent density operators and defining *exact* momenta (collective variables), we could get an *exact* canonically momenta approach to a one-dimensional (1D) neutron-proton (NP) system. In this paper, we attempt at a velocity operator approach to a 3D NP system. Following Sunakawa, after introducing momentum density operators, we define velocity operators, denoting classical fluid velocities. We derive a collective Hamiltonian in terms of the collective variables.

Keywords: Collective motion of a three-dimensional neutron-proton system; velocity operator; vortex operator; Grassmann variables

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1 Introduction

To approach elementary excitations in a Fermi system, Tomonaga gave basic idea in his collective motion theory [1, 2]. On the other hand, it is anticipated that the Sunakawa's discrete integral equation method for a Fermi system [3] may also work well for a collective motion problem. In the preceeding paper [4] (referred to as I), on *isospin* space (T, T_z) , introducing density operators $\rho_{\mathbf{k}}^{T=0, T_z=0}$ and associated variables $\pi_{\mathbf{k}}^{0,0}$ and defining *exact* momenta $\Pi_{\mathbf{k}}^{0,0}$ (collective variables), we could get an *exact* canonically momenta approach to a one-dimensional (1D) neutron-proton (NP) system. In 3D quadrupole nuclear collective motions, we also have proposed *exact* canonically momenta to collective coordinates and given *exact* canonically momenta- and collective coordinate-dependence of the kinetic part of the Hamiltonian [5]. In this paper, we attempt at a velocity operator approach to a 3D NP system. Following Sunakawa, after introducing momentum density operators $\mathbf{g}_{\mathbf{k}}^{0,0}$, we define velocity operators $\mathbf{v}_{\mathbf{k}}^{0,0}$ which denote classical fluid velocities. We derive a collective Hamiltonian in terms of the collective variables $\mathbf{v}_{\mathbf{k}}^{0,0}$ and $\rho_{\mathbf{k}}^{0,0}$ for irrotational motion. Its lowest order is diagonalized and leads us to the Bogoliubov transformation [6].

In Section 2 first we introduce collective variables $\rho_{\mathbf{k}}^{0,0}$ and associated variables $\pi_{\mathbf{k}}^{0,0}$ and give commutation relations between them. Next the velocity operator $\mathbf{v}_{\mathbf{k}}^{0,0}$ is defined by a discrete integral equation and commutation relations between the velocity operators are also given. In Section 3 the dependence of the original Hamiltonian on $\mathbf{v}_{\mathbf{k}}^{0,0}$ and $\rho_{\mathbf{k}}^{0,0}$ is determined. This section is also devoted to a calculation of a constant term in the collective Hamiltonian. Finally in Section 4 some discussions and further outlook are given.

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2 Collective variable and velocity operator

In I, we have defined the Fourier component of the density operator ($\rho(\mathbf{x}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{x})$) on the *isospin* space ($T=0, T_z=0$). In the 3D system, they are extended as

$$\rho_{\mathbf{k}}^{0,0} \equiv \frac{1}{\sqrt{A}} \sum_{\mathbf{p}, \tau_z} a_{\mathbf{p} + \frac{\mathbf{k}}{2}, \tau_z}^\dagger a_{\mathbf{p} - \frac{\mathbf{k}}{2}, \tau_z}, \quad \rho_0^{0,0} = \frac{1}{\sqrt{A}} \sum_{\mathbf{p}, \tau_z} a_{\mathbf{p}, \tau_z}^\dagger a_{\mathbf{p}, \tau_z} = \frac{N+Z}{\sqrt{A}} = \sqrt{A}, \quad (2.1)$$

where we have used the anti commutation relation (CR)s among $a_{\mathbf{k}, \tau_z}$'s and $a_{\mathbf{k}, \tau_z}^\dagger$'s given by

$$\{a_{\mathbf{k}, \tau_z}, a_{\mathbf{k}', \tau'_z}^\dagger\} = \delta_{\mathbf{k}, \mathbf{k}'} \delta_{\tau_z, \tau'_z}, \quad \{a_{\mathbf{k}, \tau_z}, a_{\mathbf{k}', \tau'_z}\} = \{a_{\mathbf{k}, \tau_z}^\dagger, a_{\mathbf{k}', \tau'_z}^\dagger\} = 0, \quad (2.2)$$

where N and Z mean the number of neutron ($\tau_z = \frac{1}{2}$) and proton ($\tau_z = -\frac{1}{2}$) [7, 8]. We here consider a spin-less fermion. The Hamiltonian H in a 3D box $\Omega (= L^3)$ is given by Eq. (2.10) in I as

$$H = T + V = \sum_{\mathbf{k}, \tau_z} \frac{\hbar^2 \mathbf{k}^2}{2m} a_{\mathbf{k}, \tau_z}^\dagger a_{\mathbf{k}, \tau_z} - \frac{A}{8\Omega} \sum_{\mathbf{k}} \nu_{T=0}(\mathbf{k}) \rho_{\mathbf{k}}^{0,0} \rho_{-\mathbf{k}}^{0,0} - \frac{A}{4\Omega} \sum_{\mathbf{k}} \nu_{T=0}(\mathbf{k}), \quad (2.3)$$

where $\nu_{T=0}$, denoted simply as ν , is a scalar function of \mathbf{k} which specifies the interaction.

Following Tomonaga [1], first we introduce associated collective momenta $\pi_{\mathbf{k}}^{0,0}$ defined by

$$\pi_{\mathbf{k}}^{0,0} \equiv \frac{m}{\mathbf{k}^2} \left(\dot{\rho}_{-\mathbf{k}}^{0,0} \right) = \frac{m}{\mathbf{k}^2} \frac{i}{\hbar} [H, \rho_{-\mathbf{k}}^{0,0}] = \frac{m}{\mathbf{k}^2} \frac{i}{\hbar} [T, \rho_{-\mathbf{k}}^{0,0}], \quad (\mathbf{k} \neq 0), \quad (2.4)$$

where the upper symbol dot $\dot{\cdot}$ means a time derivative. Calculating the commutator (2.4), we obtain the explicit expression for the associated collective variables $\pi_{\mathbf{k}}^{0,0}$ given as

$$\pi_{\mathbf{k}}^{0,0} = -\frac{i\hbar}{\sqrt{A} \mathbf{k} \cdot \mathbf{k}} \sum_{\mathbf{p}, \tau_z} \mathbf{k} \cdot \mathbf{p} a_{\mathbf{p} - \frac{\mathbf{k}}{2}, \tau_z}^\dagger a_{\mathbf{p} + \frac{\mathbf{k}}{2}, \tau_z}, \quad (\text{The symbol } \cdot \text{ denotes an inner product.}) \quad (2.5)$$

The CRs among the density $\rho_{\mathbf{k}}^{0,0}$ and the associated $\pi_{\mathbf{k}}^{0,0}$ operators, however, become as follows:

$$[\rho_{\mathbf{k}}^{0,0}, \rho_{\mathbf{k}'}^{0,0}] = 0, \quad [\pi_{\mathbf{k}}^{0,0}, \rho_{\mathbf{k}'}^{0,0}] = -\frac{i\hbar}{\sqrt{A}} \frac{\mathbf{k} \cdot \mathbf{k}'}{\mathbf{k} \cdot \mathbf{k}} \rho_{\mathbf{k}' - \mathbf{k}}^{0,0}, \quad [\pi_{\mathbf{k}}^{0,0}, \pi_{\mathbf{k}'}^{0,0}] = -\frac{i\hbar}{\sqrt{A}} \frac{\mathbf{k} \cdot \mathbf{k} - \mathbf{k}' \cdot \mathbf{k}'}{\mathbf{k} \cdot \mathbf{k}'} \pi_{\mathbf{k} + \mathbf{k}'}^{0,0}. \quad (2.6)$$

From now on, following Sunakawa [9], we introduce a momentum density operator given by

$$\mathbf{g}_{\mathbf{k}}^{0,0} \equiv \frac{\hbar}{\sqrt{A}} \sum_{\mathbf{p}, \tau_z} \mathbf{p} a_{\mathbf{p} - \frac{\mathbf{k}}{2}, \tau_z}^\dagger a_{\mathbf{p} + \frac{\mathbf{k}}{2}, \tau_z} = i\mathbf{k} \pi_{\mathbf{k}}^{0,0}, \quad \mathbf{g}_0^{0,0} = \frac{\hbar}{\sqrt{A}} \sum_{\mathbf{p}, \tau_z} \mathbf{p} a_{\mathbf{p}, \tau_z}^\dagger a_{\mathbf{p}, \tau_z} = 0. \quad (2.7)$$

In the $\mathbf{g}_0^{0,0}$, we assume zero-eigenvalue of the total momentum of our system. The CRs among $\rho_{\mathbf{k}}^{0,0}$ and vector $\mathbf{g}_{\mathbf{k}}^{0,0}$ ($= (g_{\mathbf{k}}^{0,0(1)}, g_{\mathbf{k}}^{0,0(2)}, g_{\mathbf{k}}^{0,0(3)})$) with vector \mathbf{k} ($= (k_1, k_2, k_3)$) are obtained as

$$[\mathbf{g}_{\mathbf{k}}^{0,0}, \rho_{\mathbf{k}'}^{0,0}] = \frac{\hbar \mathbf{k}'}{\sqrt{A}} \rho_{\mathbf{k}' - \mathbf{k}}^{0,0}, \quad [g_{\mathbf{k}}^{0,0(i)}, g_{\mathbf{k}'}^{0,0(j)}] = \frac{\hbar}{\sqrt{A}} (g_{\mathbf{k} + \mathbf{k}}^{0,0(i)} k_j - g_{\mathbf{k} + \mathbf{k}}^{0,0(j)} k_i). \quad (2.8)$$

Following Sunakawa [9], we define the modified momentum density operator $\mathbf{v}_{\mathbf{k}}^{0,0}$ by

$$\mathbf{v}_{\mathbf{k}}^{0,0} \equiv \mathbf{g}_{\mathbf{k}}^{0,0} - \frac{1}{\sqrt{A}} \sum_{\mathbf{p} \neq \mathbf{k}} \rho_{\mathbf{p} - \mathbf{k}}^{0,0} \mathbf{v}_{\mathbf{p}}^{0,0} \quad (\mathbf{k} \neq 0), \quad \mathbf{v}_0^{0,0} = 0. \quad (2.9)$$

Along the same way as the one in I and the Sunakawa's [3], we can prove two important CRs

$$[\mathbf{v}_{\mathbf{k}}^{0,0}, \rho_{\mathbf{k}'}^{0,0}] = \hbar \mathbf{k}' \delta_{\mathbf{k}, \mathbf{k}'}, \quad (2.10)$$

$$[v_{\mathbf{k}}^{0,0(i)}, v_{\mathbf{k}'}^{0,0(j)}] \approx -\frac{\hbar}{A} \sum_{\mathbf{p} \text{ all}} \rho_{\mathbf{p} - \mathbf{k} - \mathbf{k}'}^{0,0} (p_i v_{\mathbf{p}}^{0,0(j)} - p_j v_{\mathbf{p}}^{0,0(i)}) + \frac{1}{A} \sum'_{\mathbf{p}, \mathbf{q}} \rho_{\mathbf{p} - \mathbf{k}}^{0,0} \rho_{\mathbf{q} - \mathbf{k}'}^{0,0} [v_{\mathbf{p}}^{0,0(i)}, v_{\mathbf{q}}^{0,0(j)}]. \quad (2.11)$$

The symbol \sum' means that the term $\mathbf{p} = \mathbf{k}$ and $\mathbf{q} = \mathbf{k}'$ at the same time should be omitted. The CRs (2.10) and (2.11) are quite the same as the Sunakawa's [9]. As pointed out by him, Fourier transforms of the operators $\rho_{\mathbf{k}}^{0,0}$ and $\mathbf{v}_{\mathbf{k}}^{0,0}$ and the CRs among them are identical with those found by Landau [10] for the fluid dynamical density operator and the velocity operator. Then, it turns out that the quantum mechanical operator $\mathbf{v}(\mathbf{x})$, which satisfies the famous CR

$$[v^{0,0(i)}(\mathbf{x}), v^{0,0(j)}(\mathbf{x}')] = \frac{i\hbar}{m} \delta(\mathbf{x} - \mathbf{x}') \rho^{0,0}(\mathbf{x})^{-1} (\text{rot} \mathbf{v}^{0,0}(\mathbf{x}))^{(k)} \quad (i, j, k) \text{ cyclic}, \quad (2.12)$$

corresponds to the fluid dynamical velocity. We also have another well-known CR

$$[\mathbf{v}^{0,0}(\mathbf{x}), \rho^{0,0}(\mathbf{x}')] = -\frac{i\hbar}{m} \nabla_x \delta(\mathbf{x} - \mathbf{x}'). \quad (2.13)$$

3 $v_{\mathbf{k}}$ - and $\rho_{\mathbf{k}}$ -dependence of the Hamiltonian

We derive here a collective Hamiltonian in terms of the $\mathbf{v}_{\mathbf{k}}^{0,0}$ and $\rho_{\mathbf{k}}^{0,0}$. Following Sunakawa, we expand the kinetic operator T in a power series of the velocity operator $\mathbf{v}_{\mathbf{k}}^{0,0}$ as follows:

$$T = T_0(\rho) + \sum_{\mathbf{p} \neq 0} \mathbf{T}_1(\rho; \mathbf{p}) \cdot \mathbf{v}_{\mathbf{p}}^{0,0} + \sum_{\mathbf{p} \neq 0, \mathbf{q} \neq 0} T_2(\rho; \mathbf{p}, \mathbf{q}) \mathbf{v}_{\mathbf{p}}^{0,0} \cdot \mathbf{v}_{\mathbf{q}}^{0,0} + \dots, \quad T_2(\rho; \mathbf{p}, \mathbf{q}) = T_2(\rho; \mathbf{q}, \mathbf{p}), \quad (3.1)$$

in which $T_n (n \neq 0)$ are unknown expansion coefficients. In order to determine their explicit expressions, we take the commutators between T and $\rho_{\mathbf{k}}^{0,0}$ as follows:

$$\left. \begin{aligned} [T, \rho_{\mathbf{k}}^{0,0}] &= \hbar \mathbf{T}_1(\rho; \mathbf{k}) \cdot \mathbf{k} + 2\hbar \sum_{\mathbf{p} \neq 0} T_2(\rho; \mathbf{p}, \mathbf{k}) \mathbf{v}_{\mathbf{p}}^{0,0} \cdot \mathbf{k} + \dots, \\ [[T, \rho_{\mathbf{k}}^{0,0}], \rho_{\mathbf{k}'}^{0,0}] &= 2\hbar^2 T_2(\rho; \mathbf{k}', \mathbf{k}) \mathbf{k}' \cdot \mathbf{k} + \dots. \end{aligned} \right\} \quad (3.2)$$

On the other hand, from (2.7) and (2.9), we can calculate the commutators between T and $\rho_{\mathbf{k}}^{0,0}$ by using the relations (2.4) and (2.7) as follows:

$$[T, \rho_{\mathbf{k}}^{0,0}] = \frac{\hbar}{m} \mathbf{k} \cdot \mathbf{g}_{-\mathbf{k}}^{0,0} = \frac{\hbar}{m} \mathbf{k} \cdot \mathbf{v}_{-\mathbf{k}}^{0,0} + \frac{\hbar}{m\sqrt{A}} \sum_{\mathbf{p} \neq -\mathbf{k}} \rho_{\mathbf{p}+\mathbf{k}}^{0,0} \mathbf{k} \cdot \mathbf{v}_{\mathbf{p}}^{0,0}, \quad (3.3)$$

and using the relation (2.10) successively, we can easily obtain the following commutators:

$$[[T, \rho_{\mathbf{k}}^{0,0}], \rho_{\mathbf{k}'}^{0,0}] = -\frac{\hbar^2 \mathbf{k}^2}{m} \delta_{\mathbf{k}', -\mathbf{k}}, \quad \frac{\hbar^2}{m\sqrt{A}} \mathbf{k} \cdot \mathbf{k}' \rho_{\mathbf{k}+\mathbf{k}'}^{0,0}, \quad \mathbf{k}' \neq -\mathbf{k}, \quad (3.4)$$

$$[[[T, \rho_{\mathbf{k}}^{0,0}], \rho_{\mathbf{k}'}^{0,0}], \rho_{\mathbf{k}''}^{0,0}] = 0, \quad \dots \quad (3.5)$$

Comparing the above results with the commutators (3.2), we can determine the coefficients $T_n (n \neq 0)$. Then we can express the kinetic part T in terms of the $\rho_{\mathbf{k}}^{0,0}$ and $\mathbf{v}_{\mathbf{k}}^{0,0}$ as follows:

$$T = T_0(\rho) + \frac{1}{2m} \sum_{\mathbf{k} \neq 0} \mathbf{v}_{\mathbf{k}}^{0,0} \cdot \mathbf{v}_{-\mathbf{k}}^{0,0} + \frac{1}{2m\sqrt{A}} \sum_{\mathbf{p}+\mathbf{q} \neq 0} \rho_{\mathbf{p}+\mathbf{q}}^{0,0} \mathbf{v}_{\mathbf{p}}^{0,0} \cdot \mathbf{v}_{\mathbf{q}}^{0,0}, \quad (\mathbf{v}_0^{0,0} = 0). \quad (3.6)$$

Up to the present stage, all the expressions have been derived without any approximation.

Our remaining task is to determine the term $T_0(\rho)$ in (3.1) which depends only on $\rho_{\mathbf{k}}^{0,0}$. For this aim, we also expand it in a power series of the collective coordinates $\rho_{\mathbf{k}}^{0,0}$ in the form

$$T_0(\rho) = C_0 + \sum_{\mathbf{p} \neq 0} C_1(\mathbf{p}) \rho_{\mathbf{p}}^{0,0} + \sum_{\mathbf{p} \neq 0, \mathbf{q} \neq 0} C_2(\mathbf{p}, \mathbf{q}) \rho_{\mathbf{p}}^{0,0} \rho_{\mathbf{q}}^{0,0} + \dots, \quad C_2(\mathbf{p}, \mathbf{q}) = C_2(\mathbf{q}, \mathbf{p}), \quad (3.7)$$

and to get expansion coefficients, we take the commutators between T and $\mathbf{v}_{\mathbf{k}}^{0,0}$ as follows:

$$\left. \begin{aligned} [v_{\mathbf{k}}^{(i)}, T_0(\rho)] &= \hbar k_i C_1(\mathbf{k}) + 2\hbar k_i \sum_{\mathbf{p} \neq 0} C_2(\mathbf{p}; \mathbf{k}) \rho_{\mathbf{p}}^{0,0} + \dots, \\ [v_{\mathbf{k}'}^{(j)}, [v_{\mathbf{k}}^{(i)}, T_0(\rho)]] &= 2\hbar^2 k_i k_j' C_2(\mathbf{k}'; \mathbf{k}) + 6\hbar^2 k_i k_j' \sum_{\mathbf{p} \neq 0} C_3(\mathbf{p}; \mathbf{k}'; \mathbf{k}) \rho_{\mathbf{p}}^{0,0} + \dots. \end{aligned} \right\} \quad (3.8)$$

We here restrict our Hilbert space to subspace in which eigenvalue of the **vortex operator** satisfies $\text{rot} \mathbf{v}^{0,0}(\mathbf{x})|> = 0$, i.e., $[v_{\mathbf{k}}^{0,0(i)}, v_{\mathbf{k}'}^{0,0(j)}] = 0$. From (2.9), we have a discrete integral equation

$$[v_{\mathbf{k}}^{(i)}, T_0(\rho)] = f^{(i)}(\rho; \mathbf{k}) - \frac{1}{\sqrt{A}} \sum_{\mathbf{p} \neq 0, \mathbf{k}} \rho_{\mathbf{p}-\mathbf{k}}^{0,0} [v_{\mathbf{p}}^{(i)}, T_0(\rho)]. \quad (3.9)$$

With the aid of (3.6) and using two CRs of (2.8), the inhomogeneous term $f^{(i)}(\rho; \mathbf{k})$ becomes

$$\begin{aligned} f^{(i)}(\rho; \mathbf{k}) &\equiv [g_{\mathbf{k}}^{0,0(i)}, T_0(\rho)] = \left[g_{\mathbf{k}}^{0,0(i)}, T - \frac{1}{2m} \sum_{\mathbf{k} \neq 0} \mathbf{v}_{\mathbf{k}}^{0,0} \cdot \mathbf{v}_{-\mathbf{k}}^{0,0} - \frac{1}{2m\sqrt{A}} \sum_{\mathbf{p}+\mathbf{q} \neq 0} \rho_{\mathbf{p}+\mathbf{q}}^{0,0} \mathbf{v}_{\mathbf{p}}^{0,0} \cdot \mathbf{v}_{\mathbf{q}}^{0,0} \right] \\ &= [g_{\mathbf{k}}^{0,0(i)}, T] - \frac{\hbar}{mA} \sum_{\mathbf{p}, \mathbf{q} \text{ all}} p_i \rho_{\mathbf{p}+\mathbf{q}-\mathbf{k}}^{0,0} \mathbf{v}_{\mathbf{p}}^{0,0} \cdot \mathbf{v}_{\mathbf{q}}^{0,0} - \frac{\hbar}{mA} \sum_{\mathbf{p}, \mathbf{q} \text{ all}} \rho_{\mathbf{p}+\mathbf{q}}^{0,0} \sum_j \left(g_{\mathbf{k}+\mathbf{p}}^{0,0(i)} k_j - g_{\mathbf{k}+\mathbf{p}}^{0,0(j)} p_i \right) \mathbf{v}_{\mathbf{q}}^{0,0} \quad (3.10) \\ &= [g_{\mathbf{k}}^{0,0(i)}, T] - \frac{\hbar}{mA} \sum_{\mathbf{p}, \mathbf{q} \text{ all}} \rho_{\mathbf{p}+\mathbf{q}-\mathbf{k}}^{0,0} (\mathbf{k} \cdot \mathbf{v}_{\mathbf{p}}^{0,0}) \mathbf{v}_{\mathbf{q}}^{0,0(i)}, \quad (\text{due to change of the variable } \mathbf{p} \text{ to } \mathbf{p}-\mathbf{k}). \end{aligned}$$

To obtain the explicit formula for $f^{(i)}(\rho; \mathbf{k})$, first we calculate the CR between $g_{\mathbf{k}}^{0,0(i)}$ and T as

$$[g_{\mathbf{k}}^{0,0(i)}, T] = \frac{\hbar^3}{2m\sqrt{A}} \sum_{\mathbf{p} \text{ all}, \tau_z} p_i \left\{ \left(\mathbf{p} + \frac{\mathbf{k}}{2} \right)^2 - \left(\mathbf{p} - \frac{\mathbf{k}}{2} \right)^2 \right\} a_{\mathbf{p} - \frac{\mathbf{k}}{2}, \tau_z}^\dagger a_{\mathbf{p} + \frac{\mathbf{k}}{2}, \tau_z} = \frac{\hbar^3}{m\sqrt{A}} \sum_{\mathbf{p} \text{ all}, \tau_z} p_i (\mathbf{p} \cdot \mathbf{k}) a_{\mathbf{p} - \frac{\mathbf{k}}{2}, \tau_z}^\dagger a_{\mathbf{p} + \frac{\mathbf{k}}{2}, \tau_z}. \quad (3.11)$$

From now on, we make an approximation for $\rho_0^{0,0} = \sqrt{A}$, $v_{\mathbf{k}}^{0,0}$ and $\rho_{\mathbf{k}}^{0,0}$ as

$$v_{\mathbf{k}}^{0,0} \cong \frac{\hbar \mathbf{k}}{2} \sum_{\tau_z} \left(\bar{\theta} a_{\mathbf{k}, \tau_z} - a_{-\mathbf{k}, \tau_z}^\dagger \theta \right), \quad \rho_{\mathbf{k}}^{0,0} \cong \sum_{\tau_z} \left(\bar{\theta} a_{-\mathbf{k}, \tau_z} + a_{\mathbf{k}, \tau_z}^\dagger \theta \right). \quad (3.12)$$

The operators a_{0, τ_z} and a_{0, τ_z}^\dagger are regarded as c -numbers but with the Grassmann variables which play crucial roles to compute the inhomogeneous term $f^{(i)}(\rho; \mathbf{k})$ (3.9), as shown in I. The explicit forms of the operators are simply given as

$$a_{0, \tau_z} \cong \sqrt{A} \theta, \quad a_{0, \tau_z}^\dagger \cong \sqrt{A} \bar{\theta}, \quad (3.13)$$

where the θ and $\bar{\theta}$ are the Grassmann variables and anti-commute with $a_{\mathbf{k}, \tau_z}$ and $a_{\mathbf{k}, \tau_z}^\dagger$ [11, 12, 13]. Then the second term in the last line of (3.10) is approximately computed as

$$\begin{aligned} & -\frac{\hbar}{m\sqrt{A}} \sum_{\mathbf{p}, \mathbf{q} \text{ all}} \rho_{\mathbf{p} + \mathbf{q} - \mathbf{k}}^{0,0} (\mathbf{k} \cdot \mathbf{v}_{\mathbf{p}}^{0,0}) v_{\mathbf{q}}^{0,0(i)} \\ &= -\frac{\hbar^3}{4m\sqrt{A}} \sum_{\mathbf{p} \text{ all}} (\mathbf{p} \cdot \mathbf{k}) (k_i - p_i) \sum_{\tau_z} \left(\bar{\theta} a_{\mathbf{p}, \tau_z} - a_{-\mathbf{p}, \tau_z}^\dagger \theta \right) \sum_{\tau'_z} \left(\bar{\theta} a_{\mathbf{k} - \mathbf{p}, \tau'_z} - a_{-(\mathbf{k} - \mathbf{p}), \tau'_z}^\dagger \theta \right) \\ &= -\frac{\hbar^3}{4m\sqrt{A}} \sum_{\mathbf{p} \text{ all}} (\mathbf{p} \cdot \mathbf{k}) (k_i - p_i) \left(\rho_{-\mathbf{p}}^{0,0} - 2 \sum_{\tau_z} a_{-\mathbf{p}, \tau_z}^\dagger \theta \right) \left(\rho_{-(\mathbf{k} - \mathbf{p})}^{0,0} - 2 \sum_{\tau'_z} a_{-(\mathbf{k} - \mathbf{p}), \tau'_z}^\dagger \theta \right) \\ &= -\frac{\hbar^3}{4m\sqrt{A}} \sum_{\mathbf{p} \text{ all}} (\mathbf{p} \cdot \mathbf{k}) (k_i - p_i) \left\{ \rho_{-\mathbf{p}}^{0,0} \rho_{\mathbf{p} - \mathbf{k}}^{0,0} - 2 \sum_{\tau_z, \tau'_z} \left(\bar{\theta} \bar{\theta} a_{-\mathbf{p}, \tau_z}^\dagger a_{\mathbf{k} - \mathbf{p}, \tau'_z} + \bar{\theta} \theta a_{\mathbf{p}, \tau_z} a_{-(\mathbf{k} - \mathbf{p}), \tau'_z}^\dagger \right) \right\} \\ &= -\frac{\hbar^3}{4m\sqrt{A}} \sum_{\mathbf{p} \text{ all}} (\mathbf{p} \cdot \mathbf{k}) (k_i - p_i) \rho_{-\mathbf{p}}^{0,0} \rho_{\mathbf{p} - \mathbf{k}}^{0,0} \\ &\quad - \theta \bar{\theta} \frac{\hbar^3}{2m\sqrt{A}} \sum_{\mathbf{p} \text{ all}} \sum_{\tau_z, \tau'_z} \left\{ \left(\mathbf{p} \cdot \mathbf{k} - \frac{\mathbf{k}^2}{2} \right) \left(p_i + \frac{k_i}{2} \right) + \left(\mathbf{p} \cdot \mathbf{k} + \frac{\mathbf{k}^2}{2} \right) \left(p_i - \frac{k_i}{2} \right) \right\} a_{\mathbf{p} - \frac{\mathbf{k}}{2}, \tau_z}^\dagger a_{\mathbf{p} + \frac{\mathbf{k}}{2}, \tau'_z} \\ &= -\frac{\hbar^3}{4m\sqrt{A}} \sum_{\mathbf{p} \text{ all}} (\mathbf{p} \cdot \mathbf{k}) (k_i - p_i) \rho_{-\mathbf{p}}^{0,0} \rho_{\mathbf{p} - \mathbf{k}}^{0,0} - \theta \bar{\theta} \frac{\hbar^3}{m\sqrt{A}} \sum_{\mathbf{p} \text{ all}} \left(\mathbf{p} \cdot \mathbf{k} p_i - \frac{\mathbf{k}^2}{4} k_i \right) \sum_{\tau_z, \tau'_z} a_{\mathbf{p} - \frac{\mathbf{k}}{2}, \tau_z}^\dagger a_{\mathbf{p} + \frac{\mathbf{k}}{2}, \tau'_z} \\ &= -\frac{\hbar^3}{4m\sqrt{A}} \sum_{\mathbf{p} \text{ all}} (\mathbf{p} \cdot \mathbf{k}) (k_i - p_i) \rho_{-\mathbf{p}}^{0,0} \rho_{\mathbf{p} - \mathbf{k}}^{0,0} - \theta \bar{\theta} \frac{\hbar^3}{m} \left(\frac{\mathbf{k}}{2} \cdot \mathbf{k} \frac{k_i}{2} - \frac{\mathbf{k}^2}{4} k_i \right) \left(\bar{\theta} a_{\mathbf{k}, \tau_z} + a_{-\mathbf{k}}^\dagger \theta \right), \end{aligned} \quad (3.14)$$

the last term of which is obtained by extracting the terms with $\mathbf{p} = \frac{\mathbf{k}}{2}$ or $\mathbf{p} = -\frac{\mathbf{k}}{2}$ in the last term in the second line from the bottom and it evidently vanishes.

Substituting the resultant formula of (3.11), $[g_{\mathbf{k}}^{0,0(i)}, T] = \frac{\hbar^3 k_i \mathbf{k}^2}{4m} \rho_{-\mathbf{k}}^{0,0}$, and the calculated result (3.14) into $f^{(i)}(\rho; \mathbf{k})$, i.e., (3.10), we get an approximate formula for the $f^{(i)}(\rho; \mathbf{k})$ up to the order of $\frac{1}{\sqrt{A}}$ in the following form:

$$f^{(i)}(\rho; \mathbf{k}) = \frac{\hbar^3 k_i \mathbf{k}^2}{4m} \rho_{-\mathbf{k}}^{0,0} - \frac{\hbar^3}{4m\sqrt{A}} \sum_{\mathbf{p} \neq \mathbf{k}} (\mathbf{p} \cdot \mathbf{k}) (k_i - p_i) \rho_{-\mathbf{p}}^{0,0} \rho_{\mathbf{p} - \mathbf{k}}^{0,0}. \quad (3.15)$$

Further substituting (3.15) into (3.9) and bringing the next leading term, we can rewrite the R.H.S. of the discrete integral equation (3.9) as

$$\begin{aligned} [v_{\mathbf{k}}^{0,0(i)}, T_0(\rho)] &= \frac{\hbar^3 k_i \mathbf{k}^2}{4m} \rho_{-\mathbf{k}}^{0,0} - \frac{\hbar^3}{4m\sqrt{A}} \sum_{\mathbf{p} \neq \mathbf{k}} \{ (\mathbf{p} \cdot \mathbf{k}) (k_i - p_i) + p_i \mathbf{p}^2 \} \rho_{-\mathbf{p}}^{0,0} \rho_{\mathbf{p} - \mathbf{k}}^{0,0} \\ &\approx \frac{\hbar^3 k_i \mathbf{k}^2}{4m} \rho_{-\mathbf{k}}^{0,0} - \frac{\hbar^3 k_i}{8m\sqrt{A}} \sum_{\mathbf{p} \neq \mathbf{k}} \mathbf{p} \cdot (\mathbf{p} + \mathbf{k}) \rho_{-\mathbf{p}}^{0,0} \rho_{\mathbf{p} - \mathbf{k}}^{0,0}, \quad \left(\text{under the assumption of } p_i = \frac{k_i}{2} \right). \end{aligned} \quad (3.16)$$

From (3.16) and the CRs (2.6) and (2.10), we get the following commutation relations:

$$\left. \begin{aligned} [v_{\mathbf{k}'}^{0,0(j)}, [v_{\mathbf{k}}^{0,0(i)}, T_0(\rho)]] &= -\frac{\hbar^4 k_i k_j \mathbf{k}^2}{4m} \delta_{\mathbf{k}', -\mathbf{k}} - \frac{\hbar^4 k_i k_j'}{4m\sqrt{A}} (\mathbf{k}^2 + \mathbf{k} \cdot \mathbf{k}' + \mathbf{k}'^2) \rho_{-\mathbf{k}-\mathbf{k}'}^{0,0}, \\ [v_{\mathbf{k}''}^{0,0(k)}, [v_{\mathbf{k}'}^{0,0(j)}, [v_{\mathbf{k}}^{0,0(i)}, T_0(\rho)]]] &= -\frac{\hbar^5 k_i k_j' (k_k + k_k')}{4m\sqrt{A}} (\mathbf{k}^2 + \mathbf{k} \cdot \mathbf{k}' + \mathbf{k}'^2), \\ [v_{\mathbf{k}'''}^{0,0(l)}, [v_{\mathbf{k}''}^{0,0(k)}, [v_{\mathbf{k}'}^{0,0(j)}, [v_{\mathbf{k}}^{0,0(i)}, T_0(\rho)]]]] &= 0. \end{aligned} \right\} \quad (3.17)$$

By a procedure similar to the previous one, we can determine the coefficients $C_n (n \neq 0)$ in (3.7) and then get an approximate form of $T_0(\rho)$ in terms of variables $\rho_{\mathbf{k}}^{0,0}$ in the following form:

$$\begin{aligned} T_0(\rho) &= C_0 + \frac{\hbar^2}{8m} \sum_{\mathbf{k} \neq 0} \mathbf{k}^2 \rho_{\mathbf{k}}^{0,0} \rho_{-\mathbf{k}}^{0,0} \\ &\quad - \frac{\hbar^2}{24m\sqrt{A}} \sum_{\mathbf{p} \neq 0, \mathbf{q} \neq 0, \mathbf{p}+\mathbf{q} \neq 0} (\mathbf{p}^2 + \mathbf{p} \cdot \mathbf{q} + \mathbf{q}^2) \rho_{\mathbf{p}}^{0,0} \rho_{\mathbf{q}}^{0,0} \rho_{-\mathbf{p}-\mathbf{q}}^{0,0} + O\left(\frac{1}{A}\right). \end{aligned} \quad (3.18)$$

With the aid of the underlying identities

$$\left. \begin{aligned} \sum_{\mathbf{p} \neq 0, \mathbf{q} \neq 0, \mathbf{p}+\mathbf{q} \neq 0} \mathbf{p}^2 \rho_{\mathbf{p}}^{0,0} \rho_{\mathbf{q}}^{0,0} \rho_{-\mathbf{p}-\mathbf{q}}^{0,0} &= \sum_{\mathbf{p} \neq 0, \mathbf{q} \neq 0, \mathbf{p}+\mathbf{q} \neq 0} (\mathbf{p}+\mathbf{q})^2 \rho_{\mathbf{p}}^{0,0} \rho_{\mathbf{q}}^{0,0} \rho_{-\mathbf{p}-\mathbf{q}}^{0,0}, \\ \sum_{\mathbf{p} \neq 0, \mathbf{q} \neq 0, \mathbf{p}+\mathbf{q} \neq 0} \mathbf{p}^2 \rho_{\mathbf{p}}^{0,0} \rho_{\mathbf{q}}^{0,0} \rho_{-\mathbf{p}-\mathbf{q}}^{0,0} &= -2 \sum_{\mathbf{p} \neq 0, \mathbf{q} \neq 0, \mathbf{p}+\mathbf{q} \neq 0} \mathbf{p} \cdot \mathbf{q} \rho_{\mathbf{p}}^{0,0} \rho_{\mathbf{q}}^{0,0} \rho_{-\mathbf{p}-\mathbf{q}}^{0,0}, \end{aligned} \right\} \quad (3.19)$$

the lowest kinetic term of T , $T_0(\rho)$ (3.18) is rewritten as

$$T_0(\rho) = C_0 + \frac{\hbar^2}{8m} \sum_{\mathbf{k} \neq 0} \mathbf{k}^2 \rho_{\mathbf{k}}^{0,0} \rho_{-\mathbf{k}}^{0,0} + \frac{\hbar^2}{8m\sqrt{A}} \sum_{\mathbf{p} \neq 0, \mathbf{q} \neq 0, \mathbf{p}+\mathbf{q} \neq 0} \mathbf{p} \cdot \mathbf{q} \rho_{\mathbf{p}}^{0,0} \rho_{\mathbf{q}}^{0,0} \rho_{-\mathbf{p}-\mathbf{q}}^{0,0}. \quad (3.20)$$

From now on, we calculate the constant term C_0 , the first term in the R.H.S. of (3.18). Substituting (3.18) into (3.6), the constant term C_0 is computed up to the order of $\frac{1}{A}$:

$$\begin{aligned} C_0 &= T - \frac{\hbar^2}{8m} \sum_{\mathbf{k} \neq 0} \mathbf{k}^2 \rho_{\mathbf{k}}^{0,0} \rho_{-\mathbf{k}}^{0,0} - \frac{1}{2m} \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}}^{0,0} \cdot \mathbf{v}_{-\mathbf{k}}^{0,0} + \frac{1}{2m\sqrt{A}} \sum_{\mathbf{p}+\mathbf{q} \neq 0} \rho_{\mathbf{p}+\mathbf{q}}^{0,0} \mathbf{v}_{\mathbf{p}}^{0,0} \cdot \mathbf{v}_{\mathbf{q}}^{0,0} \\ &\quad - \frac{\hbar^2}{8m\sqrt{A}} \sum_{\mathbf{p} \neq 0, \mathbf{q} \neq 0, \mathbf{p}+\mathbf{q} \neq 0} \mathbf{p} \cdot \mathbf{q} \rho_{\mathbf{p}}^{0,0} \rho_{\mathbf{q}}^{0,0} \rho_{-\mathbf{p}-\mathbf{q}}^{0,0}. \end{aligned} \quad (3.21)$$

Using $\rho_{\mathbf{k}}^{0,0} \cong \sum_{\tau_z} (\bar{\theta} a_{-\mathbf{k}, \tau_z} + a_{\mathbf{k}, \tau_z}^\dagger \theta)$ and $\mathbf{v}_{\mathbf{k}}^{0,0} \cong \frac{\hbar \mathbf{k}}{2} \sum_{\tau_z} (\bar{\theta} a_{\mathbf{k}, \tau_z} - a_{-\mathbf{k}, \tau_z}^\dagger \theta)$, we can calculate the third term in (3.21) very similarly to the calculation of (3.14) and then reach to a result such as

$$\begin{aligned} -\frac{1}{2m} \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}}^{0,0} \cdot \mathbf{v}_{-\mathbf{k}}^{0,0} &= \frac{\hbar^2}{8m} \sum_{\mathbf{k}} \mathbf{k}^2 \sum_{\tau_z} (\bar{\theta} a_{\mathbf{k}, \tau_z} - a_{-\mathbf{k}, \tau_z}^\dagger \theta) \sum_{\tau_z'} (\bar{\theta} a_{-\mathbf{k}, \tau_z'} - a_{\mathbf{k}, \tau_z'}^\dagger \theta) \\ &= \frac{\hbar^2}{8m} \sum_{\mathbf{k}} \mathbf{k}^2 \left(\rho_{-\mathbf{k}}^{0,0} - 2 \sum_{\tau_z} a_{-\mathbf{k}, \tau_z}^\dagger \theta \right) \left(\rho_{\mathbf{k}}^{0,0} - 2 \sum_{\tau_z'} a_{\mathbf{k}, \tau_z'}^\dagger \theta \right) \\ &= \frac{\hbar^2}{8m} \sum_{\mathbf{k}} \mathbf{k}^2 \rho_{\mathbf{k}}^{0,0} \rho_{-\mathbf{k}}^{0,0} - \theta \bar{\theta} \sum_{\mathbf{k}} \frac{\hbar^2 \mathbf{k}^2}{2m} \sum_{\tau_z, \tau_z'} a_{\mathbf{k}, \tau_z}^\dagger a_{\mathbf{k}, \tau_z'} + \theta \bar{\theta} \frac{\hbar^2}{2m} \sum_{\mathbf{k}} \mathbf{k}^2 + O\left(\frac{1}{A}\right). \end{aligned} \quad (3.22)$$

As for the forth and last terms in (3.21), due to the relations $\theta\theta=0$ and $\bar{\theta}\bar{\theta}=0$, we simply have

$$\begin{aligned} &\frac{1}{2m\sqrt{A}} \sum_{\mathbf{p}+\mathbf{q} \neq 0} \rho_{\mathbf{p}+\mathbf{q}}^{0,0} \mathbf{v}_{\mathbf{p}}^{0,0} \cdot \mathbf{v}_{\mathbf{q}}^{0,0} \\ &\cong \frac{\hbar^2}{8m\sqrt{A}} \sum_{\mathbf{p}+\mathbf{q} \neq 0} \mathbf{p} \cdot \mathbf{q} \sum_{\tau_z} (\bar{\theta} a_{-\mathbf{p}-\mathbf{q}, \tau_z} + a_{\mathbf{p}+\mathbf{q}, \tau_z}^\dagger \theta) \sum_{\tau_z'} (\bar{\theta} a_{\mathbf{p}, \tau_z'} - a_{-\mathbf{p}, \tau_z'}^\dagger \theta) \sum_{\tau_z''} (\bar{\theta} a_{\mathbf{q}, \tau_z''} - a_{-\mathbf{q}, \tau_z''}^\dagger \theta) \\ &= 0 \text{ and also} \end{aligned} \quad (3.23)$$

$$\begin{aligned} &-\frac{\hbar^2}{8m\sqrt{A}} \sum_{\mathbf{p} \neq 0, \mathbf{q} \neq 0, \mathbf{p}+\mathbf{q} \neq 0} \mathbf{p} \cdot \mathbf{q} \rho_{\mathbf{p}}^{0,0} \rho_{\mathbf{q}}^{0,0} \rho_{-\mathbf{p}-\mathbf{q}}^{0,0} = 0. \text{ Substituting these into (3.21), we have} \\ &C_0 \cong (1 - \theta\bar{\theta}) T + \theta\bar{\theta} \frac{\hbar^2}{2m} \sum_{\mathbf{k}} \mathbf{k}^2 - \theta\bar{\theta} \sum_{\mathbf{k}, \tau_z \neq \tau_z'} \frac{\hbar^2 \mathbf{k}^2}{2m} a_{\mathbf{k}, \tau_z}^\dagger a_{\mathbf{k}, \tau_z'} \cong \frac{\hbar^2}{2m} \sum_{\mathbf{k}} \mathbf{k}^2. \end{aligned} \quad (3.24)$$

Here we have used the relation $\theta\bar{\theta}=1$ and neglected the term $\sum_{\mathbf{k}, \tau_z \neq \tau'_z} \frac{\hbar^2 \mathbf{k}^2}{2m} a_{\mathbf{k}, \tau_z}^\dagger a_{\mathbf{k}, \tau'_z}$ which does not exist in the case of the *isospin*-less Fermion system. The result (3.24) is not identical with the Sunakawa's result [3] for a Bose system. This is because we have dealt with a Fermi system. Then we get a result which is considered as a natural consequence for a Fermi system. It is surprising to see that the C_0 (3.24) coincides with the constant term in the resultant ground state energy given by the Tomonaga's method [1]. Using (3.6), (3.18) and (3.24) and separating the term $C_0 \cong \sum_{\mathbf{k}} \frac{\hbar^2 \mathbf{k}^2}{2m}$ into two parts $-\sum_{\mathbf{k}} \frac{\hbar^2 \mathbf{k}^2}{4m}$ and $\frac{3}{2} \sum_{\mathbf{k}} \frac{\hbar^2 \mathbf{k}^2}{2m}$, we reach final goal of expressing the Hamiltonian H (2.3) in terms of $\rho_{\mathbf{k}}^{0,0}$ and $\mathbf{v}_{\mathbf{k}}^{0,0}$ as follows:

$$H = -\frac{A(A+2)}{8\Omega} \nu(0) - \frac{A}{4\Omega} \sum_{\mathbf{k} \neq 0} \nu(\mathbf{k}) - \sum_{\mathbf{k}} \frac{\hbar^2 \mathbf{k}^2}{4m} + \frac{3}{2} \sum_{\mathbf{k}} \frac{\hbar^2 \mathbf{k}^2}{2m} \\ + \sum_{\mathbf{k} \neq 0} \left\{ \frac{1}{2m} \mathbf{v}_{\mathbf{k}}^{0,0} \cdot \mathbf{v}_{-\mathbf{k}}^{0,0} + \left(\frac{\hbar^2 \mathbf{k}^2}{8m} - \frac{A}{8\Omega} \nu(\mathbf{k}) \right) \rho_{\mathbf{k}}^{0,0} \rho_{-\mathbf{k}}^{0,0} \right\} \\ - \frac{1}{2m\sqrt{A}} \sum_{\mathbf{p} \neq 0, \mathbf{q} \neq 0, \mathbf{p}+\mathbf{q} \neq 0} \mathbf{p} \cdot \mathbf{q} \rho_{\mathbf{p}+\mathbf{q}}^{0,0} \mathbf{v}_{\mathbf{p}}^{0,0} \mathbf{v}_{\mathbf{q}}^{0,0} + \frac{1}{8m\sqrt{A}} \sum_{\mathbf{p} \neq 0, \mathbf{q} \neq 0, \mathbf{p}+\mathbf{q} \neq 0} \mathbf{p} \cdot \mathbf{q} \rho_{\mathbf{p}}^{0,0} \rho_{\mathbf{q}}^{0,0} \rho_{-\mathbf{p}-\mathbf{q}}^{0,0}, \quad (3.25)$$

in which the standard expression for the interaction V in *isospin* $T=0$ nuclei is given by (2.9) and (2.10) in I. In the case of 3D nuclei, there appear volume Ω instead of length L . Then we have the terms such as $-\frac{A}{4\Omega}$ and $-\frac{A}{8\Omega}$. The expression (3.25) is just the Sunakawa's form up to the order of $\frac{1}{\sqrt{A}}$ [3], except the last term $\frac{3}{2} \sum_{\mathbf{k}} \frac{\hbar^2 \mathbf{k}^2}{2m}$ in the first line of the R.H.S. of (3.25). The second term $-\frac{A}{4\Omega} \nu(\mathbf{k})$ also in the first line is separated into two parts $\frac{A}{8\Omega} \nu(\mathbf{k})$ and $-\frac{3A}{8\Omega} \nu(\mathbf{k})$. These differences arise due to the fact that we deal with a *isospin* $T=0$ Fermi system but not a Bose system. At the present moment, we discard the underlined term. In (3.25), the sum of some terms below are considered as the lowest order Hamiltonian H_0 ,

$$H_0 = -\frac{A(A+2)}{8\Omega} \nu(0) + \sum_{\mathbf{k} \neq 0} \left\{ -\frac{\hbar^2 \mathbf{k}^2}{4m} + \frac{A}{8\Omega} \nu(\mathbf{k}) + \frac{1}{2m} \mathbf{v}_{\mathbf{k}}^{0,0} \cdot \mathbf{v}_{-\mathbf{k}}^{0,0} + \left(\frac{\hbar^2 \mathbf{k}^2}{8m} - \frac{A}{8\Omega} \nu(\mathbf{k}) \right) \rho_{\mathbf{k}}^{0,0} \rho_{-\mathbf{k}}^{0,0} \right\}. \quad (3.26)$$

Now, let us introduce the Boson annihilation and creation operators defined as

$$\alpha_{\mathbf{k}} \equiv \sqrt{\frac{mE_{\mathbf{k}}}{2\hbar^2 \mathbf{k}^2}} \rho_{-\mathbf{k}}^{0,0} + \frac{1}{\sqrt{2m\mathbf{k}^2 E_{\mathbf{k}}}} \mathbf{k} \cdot \mathbf{v}_{\mathbf{k}}^{0,0}, \quad \alpha_{\mathbf{k}}^\dagger \equiv \sqrt{\frac{mE_{\mathbf{k}}}{2\hbar^2 \mathbf{k}^2}} \rho_{\mathbf{k}}^{0,0} + \frac{1}{\sqrt{2m\mathbf{k}^2 E_{\mathbf{k}}}} \mathbf{k} \cdot \mathbf{v}_{-\mathbf{k}}^{0,0}, \quad (\mathbf{k} \neq 0). \quad (3.27)$$

Using (3.27) and (3.12), the collective variables $\rho_{\mathbf{k}}^{0,0}$ and $\mathbf{v}_{\mathbf{k}}^{0,0}$ are expressed as

$$\left. \begin{aligned} \rho_{\mathbf{k}}^{0,0} &= \sqrt{\frac{\hbar^2 \mathbf{k}^2}{2mE_{\mathbf{k}}}} \frac{1}{2} (\alpha_{-\mathbf{k}} + \alpha_{\mathbf{k}}^\dagger) = \sum_{\tau_z} \bar{\theta} a_{-\mathbf{k}, \tau_z} + \sum_{\tau_z} a_{\mathbf{k}, \tau_z}^\dagger \theta, \quad (\mathbf{k} \neq 0), \\ \mathbf{v}_{\mathbf{k}}^{0,0} &= -i \frac{\sqrt{2mE_{\mathbf{k}}}}{\mathbf{k} \cdot \mathbf{k}} \frac{\mathbf{k}}{2} (\alpha_{\mathbf{k}} - \alpha_{-\mathbf{k}}^\dagger) = \frac{\hbar \mathbf{k}}{2} (\sum_{\tau_z} \bar{\theta} a_{\mathbf{k}, \tau_z} - \sum_{\tau_z} a_{-\mathbf{k}, \tau_z}^\dagger \theta), \quad (\mathbf{k} \neq 0), \end{aligned} \right\} \quad (3.28)$$

and substituting which into (3.26), the lowest order Hamiltonian H_0 (3.26) is diagonalized as

$$H_0 = E_0^G + \sum_{\mathbf{k} \neq 0} E_{\mathbf{k}} \alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}}, \quad E_0^G \equiv -\frac{A(A+2)}{8\Omega} \nu(0) - \frac{1}{2} \sum_{\mathbf{k} \neq 0} \left(E_{\mathbf{k}} - \frac{\hbar^2 \mathbf{k}^2}{2m} - \frac{A}{2\Omega} \nu(\mathbf{k}) \right), \quad (3.29) \\ E_{\mathbf{k}} \equiv \sqrt{(\varepsilon_{\mathbf{k}})^2 - \frac{\hbar^2 \mathbf{k}^2}{m} \frac{A}{4\Omega} \nu(\mathbf{k})}, \quad \varepsilon_{\mathbf{k}} \equiv \frac{\hbar^2 \mathbf{k}^2}{2m}, \quad (E_{\mathbf{k}}: \text{Quasi-particle energy}),$$

where we have used the commutation relation $[\mathbf{v}_{\mathbf{k}}^{0,0}, \rho_{\mathbf{k}}^{0,0}] = \hbar \mathbf{k}$ given by (2.10). In this sense, the zero point energy of the collective mode is included in the above diagonalization. Since $E_{\mathbf{k}}$ is approximated as $\frac{\hbar^2 \mathbf{k}^2}{2m} - \frac{A}{4\Omega} \nu(\mathbf{k})$, the term $\frac{1}{2} (E_{\mathbf{k}} - \frac{\hbar^2 \mathbf{k}^2}{2m} - \frac{A}{2\Omega} \nu(\mathbf{k}))$ becomes $-\frac{3A}{8\Omega} \nu(\mathbf{k})$.

This is why we must separate $-\frac{A}{4\Omega}\nu(\mathbf{k})$ into $\frac{A}{8\Omega}\nu(\mathbf{k})$ and $-\frac{3A}{8\Omega}\nu(\mathbf{k})$ but not arbitrary. These facts lead us to H_0 (3.26). The quantity E_0^G in (3.29) corresponds to the ground state energy. Thus we have a Bogoliubov transformation for Boson-like operators $\sum_{\tau_z} \bar{\theta} a_{\mathbf{k},\tau_z}$ and $\sum_{\tau_z} a_{\mathbf{k},\tau_z}^\dagger \theta$ as

$$\alpha_{\mathbf{k}} = \frac{(E_{\mathbf{k}+\varepsilon\mathbf{k}}) \sum_{\tau_z} \bar{\theta} a_{\mathbf{k},\tau_z} + (E_{\mathbf{k}-\varepsilon\mathbf{k}}) \sum_{\tau_z} a_{-\mathbf{k},\tau_z}^\dagger \theta}{2\sqrt{\varepsilon_{\mathbf{k}} E_{\mathbf{k}}}}, \quad \alpha_{-\mathbf{k}}^\dagger = \frac{(E_{\mathbf{k}-\varepsilon\mathbf{k}}) \sum_{\tau_z} \bar{\theta} a_{\mathbf{k},\tau_z} + (E_{\mathbf{k}+\varepsilon\mathbf{k}}) \sum_{\tau_z} a_{-\mathbf{k},\tau_z}^\dagger \theta}{2\sqrt{\varepsilon_{\mathbf{k}} E_{\mathbf{k}}}}, \quad (3.30)$$

which is the same as the famous Bogoliubov transformation for the usual Bosons [6]. The diagonalization (3.29) has been given similarly to the usual Bose system by Sunakawa [9].

4 Discussions and further outlook

In the preceding sections, we have proposed a velocity operator approach to a 3D NP system. After introducing collective variables, the velocity operator approach to the 3D NP system could be provided. Particularly, an interesting problem of describing excitations occurring in nuclei, *isospin* $T = 0$ surface vibrations, may be possible to treat as an elementary exercise. For this problem, for example, see textbooks [14, 15]. By applying the velocity operator approach to such a problem, an excellent description of the excitations in *isospin* $T = 0$ nuclei will be expected to reproduce various correct behaviors including excited energies. Because the present theory is constructed to take into account important many-body correlations, which have not been investigated sufficiently for a long time in the historical ways for such a problem. In this context, it is said that a new field of exploration of excitations in a 3D Fermi system may open with aid of the velocity operator approach whose new achievement may be appeared elsewhere. By the way, connection of the present theory with the fluid dynamics was been mentioned briefly by Sunakawa. He transformed the quantum-fluid Hamiltonian (3.25) to the one in the configuration space and obtained the classical-fluid Hamiltonian for the case of the irrotational flow [16, 9].

There also exist *isospin* $T=1$ excitations in nuclei. As stressed in I, the structures of the commutators among $\rho_{\mathbf{k}}^{0,0}, \rho_{\mathbf{k}}^{1,0}, \mathbf{g}_{\mathbf{k}}^{0,0}(=i\mathbf{k}\pi_{\mathbf{k}}^{0,0})$ and $\mathbf{g}_{\mathbf{k}}^{1,0}(=i\mathbf{k}\pi_{\mathbf{k}}^{1,0})$ have the twisted property in the *isospin* space (T, T_z) . Due to this fact, commutators $[\mathbf{g}_{\mathbf{k}}^{1,0}, \rho_{\mathbf{k}'}^{1,0}]$ and $[\mathbf{g}_{\mathbf{k}}^{1,0}, \mathbf{g}_{\mathbf{k}'}^{1,0}]$ are not closed. The velocity operator $\mathbf{v}_{\mathbf{k}}^{1,0}$ defined in the same way as (2.9) and density operator $\rho_{\mathbf{k}}^{1,0}$ do not satisfy an important commutator $[\mathbf{v}_{\mathbf{k}}^{1,0}, \rho_{\mathbf{k}'}^{1,0}] = \hbar \mathbf{k}' \delta_{\mathbf{k}\mathbf{k}'}$. Therefore, the $\rho_{\mathbf{k}}^{1,0}$ and $\mathbf{v}_{\mathbf{k}}^{1,0}$ are not a suitable pair of collective operators for our object. Then it turns out that the isovector $T=1$ surface vibrations [14, 15] can't be treated in the present approach.

As described in Section 3, hitherto, we have restricted Hilbert space to subspace in which the **vortex operator** satisfies $\text{rot} \mathbf{v}^{0,0}(\mathbf{x})| > 0$, i.e., $[v_{\mathbf{k}}^{0,0(i)}, v_{\mathbf{k}'}^{0,0(j)}] = 0$. While, in the classical fluid dynamics, the velocity field $\mathbf{v}(\mathbf{x})$ is given as $\mathbf{v}(\mathbf{x}) = -\nabla\phi(\mathbf{x}) - \lambda(\mathbf{x})\nabla\psi(\mathbf{x})$, where $\phi(\mathbf{x})$ is the velocity potential and $\lambda(\mathbf{x})$ and $\psi(\mathbf{x})$ are Clebsch parameters [17]. This fact was already been pointed out by Sunakawa [3]. As was suggested long time ago by Marumori *et al.* [18] and Watanabe [19], the **internal rotational** motion, i.e., the vortex motion, however, may exist also in nuclei. So, we have something worthwhile in taking the vortex motion into consideration. In the very near future, we will attempt at a description of **rotational** velocity field of a fluid in nuclei through a Clebsch transformation. Contrary to the above ways to the vortex motion in nuclei, we should notice the paper in which Holtzwarth and Schütte have attempted at a derivation of fluid-dynamical equations of motion which allow for velocity fields with vorticity, starting from a time-dependent variational principle for a many-fermion system. They have derived an interesting relation between the vorticity and the two-body correlations [20]. Standing on the above Clebsch viewpoint and Ziman's [21], going from classical fluid dynamics to quantum fluid dynamics, we will derive a **vortex** Hamiltonian of the fluid in terms of roton operators. The quantum fluid-dynamical approach may be applied to a realistic nuclei. Such an application to nuclei will provide an excellent description of another kind of elementary energy excitation, so-called the "**vortex excitation**" occurred in nuclei because the quantum fluid-dynamical manner may approach various features of many-body effects, which have been discarded in the traditional treatments of the problem of rotational collective motion. This work will be presented elsewhere in a forthcoming paper.

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